Finite Biorthogonal Transforms and Multiresolution Analyses on Intervals

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Multiresolution Analysis

Definition

A *Multiresolution Analysis* (MRA) of $L^2(\mathbb{R})$ is an infinite nested sequence of linear subspaces of $L^2(\mathbb{R})$

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$$

with the properties

(i) $\bigcup_n V_n$ is dense in $L^2(\mathbb{R})$, $\quad \bigcap_n V_n = \{0\}$

(ii) $f(x) \in V_n \iff f(2x) \in V_{n+1}$ for all n $\in \mathbb{Z}$

(iii) $f(x) \in V_n \iff f(x - 2^{-n}k) \in V_n$ for all n, k $\in \mathbb{Z}$

(iv) There exists ϕ such that $\{\phi(x-k) : k \in \mathbb{Z}\}$ forms a Riesz basis of V_0

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$$\phi(x) = \sum_k s_k \phi(2x - k)$$

$$\{\phi_{n,k} = 2^{\frac{n}{2}} \phi(2^n x - k) : k \in \mathbb{Z}\} \text{ forms a Riesz basis for } V_n.$$

Biorthogonality

Suppose (real or complex) scaling coefficients $\{s_k\}$, $\{\tilde{s}_k\}$ are given such that ϕ and $\tilde{\phi}$ are two compactly supported scaling functions in $L^2(\mathbb{R})$

$$\phi(x) = \sqrt{2} \sum_k s_k \phi(2x - k), \quad \tilde{\phi}(x) = \sqrt{2} \sum_k \tilde{s}_k \tilde{\phi}(2x - k),$$

which are biorthogonal

$$\left\langle \phi(x), \tilde{\phi}(x - n) \right\rangle = \delta_{0n},$$

and whose integer shifts form Riesz bases for two subspaces of $L^2(\mathbb{R})$,

$$V_0 = \text{Linear Span of } \{\phi(x - k) : k \in \mathbb{Z}\}$$

$$\widetilde{V}_0 = \text{Linear Span of } \{\tilde{\phi}(x - k) : k \in \mathbb{Z}\}.$$

Projection Operators

$$P_n(f) = \sum_k \langle f(x), \tilde{\phi}_{n,k}(x) \rangle \phi_{n,k}(x), \quad \widetilde{P}_n(f) = \sum_k \langle f(x), \phi_{n,k}(x) \rangle \tilde{\phi}_{n,k}(x).$$

Letting $Q_n = P_{n+1} - P_n$ and $\widetilde{Q}_n = \widetilde{P}_{n+1} - \widetilde{P}_n$, we define the spaces W_n and $\widetilde{W}_n$ as the range of Q_n and $\widetilde{Q}_n$, respectively. These spaces satisfy

$$V_n \bigoplus W_n = V_{n+1}, \quad \widetilde{V}_n \bigoplus \widetilde{W}_n = \widetilde{V}_{n+1}.$$

These are not necessarily orthogonal sums, however

$$W_n \perp \widetilde{V}_n, \quad \widetilde{W}_n \perp V_n.$$

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$$L^2(\mathbb{R}) = \bigoplus W_n = \bigoplus \widetilde{W}_n$$

Wavelet Spaces

$$W_n = \text{The linear span of } \{2^{n/2}\psi(2^n x - k) : k \in \mathbb{Z}\},$$

$$\widetilde{W}_n = \text{The linear span of } \{2^{n/2}\tilde{\psi}(2^n x - k) : k \in \mathbb{Z}\}.$$

where

$$\psi(x) = \sqrt{2}\sum_k w_k \phi(2x - k), \quad \tilde{\psi}(x) = \sqrt{2}\sum_k \tilde{w}_k \tilde{\phi}(2x - k)$$

and

$$w_k = (-1)^k \tilde{s}_{N-k}, \quad \tilde{w}_k = (-1)^k s_{N-k}, \quad N \text{ odd.}$$

Encoding

Given an ℓ^2 sequence $f = \{f_k\}$, we can use the scaling and wavelet coefficients to decompose f into low and high frequency components. i.e.

$$\begin{pmatrix} \ell \\ h \end{pmatrix} = \begin{pmatrix} S \\ W \end{pmatrix} f$$

where $S_{j,k} = s_{j-2k}$, $W_{j,k} = w_{j-2k}$.

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e.g.

$$\begin{pmatrix} \vdots \\ \ell_0 \\ \ell_1 \\ \ell_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} & \ddots & & & & & & \\ \dots & s_0 & s_1 & s_2 & s_3 & 0 & 0 & \dots \\ \dots & 0 & 0 & s_0 & s_1 & s_2 & s_3 & \dots \\ \dots & 0 & 0 & 0 & 0 & s_0 & s_1 & \dots \\ & & & & \ddots & & & \end{pmatrix} \begin{pmatrix} \vdots \\ f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}$$

Reconstruction

$$\left\langle \phi(x), \tilde{\phi}(x-n) \right\rangle = \delta_{0n}$$

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$$\sum_k s_k \tilde{s}_{k-2j} = \sum_k w_k \tilde{w}_{k-2j} = \delta_{0j},$$

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which provides exact reconstruction. If

$$\begin{pmatrix} \ell \\ h \end{pmatrix} = \begin{pmatrix} S \\ W \end{pmatrix} f,$$

then

$$\begin{pmatrix} \widetilde{S} \\ \widetilde{W} \end{pmatrix}^* \begin{pmatrix} \ell \\ h \end{pmatrix} = \begin{pmatrix} \widetilde{S} \\ \widetilde{W} \end{pmatrix}^* \begin{pmatrix} S \\ W \end{pmatrix} f = f.$$

Finite Discrete Wavelet Transforms

To apply this theory to a finite length $f = \{f_k\}_{k=0}^{2n_f-1}$ a natural approach would be to first *periodize* the data. The resulting decompositions ℓ and h are then periodic.

Considering one period of ℓ and h is equivalent to applying certain finite matrices to the original finite signal f. (For convenience we again denote these matrices S, W, $\widetilde{S}$, and $\widetilde{W}$.) For instance,

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_4 & s_5 & 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 & s_5 & 0 & 0 & 0 & 0 & s_0 & s_1 \end{pmatrix}.$$

However this transformation will intertwine the data at the beginning and end of f, which is undesirable.

Let $T = \begin{pmatrix} S \\ W \end{pmatrix}$, $\widetilde{T} = \begin{pmatrix} \widetilde{S} \\ \widetilde{W} \end{pmatrix}$, and notice that the pair of matrices $\left(T, \widetilde{T}\right)$ are *biorthogonal*, i.e. $T\widetilde{T}^* = I$.

Theorem

Given the matrices T and $\widetilde{T}$ there exist biorthogonal pairs of matrices $\left(U, \widetilde{U}\right)$ and $\left(V, \widetilde{V}\right)$ such that $Q = \widetilde{U}T\widetilde{V}$, $\widetilde{Q} = U\widetilde{T}V$ are biorthogonal and consist of banded block matrices.

Letting $Q = \begin{pmatrix} M \\ N \end{pmatrix}$, $\widetilde{Q} = \begin{pmatrix} \widetilde{M} \\ \widetilde{N} \end{pmatrix}$, the banded matrices M and $\widetilde{M}$ play the role that S and $\widetilde{S}$ did before.

$$M = \begin{pmatrix} A_0 & A_1 & A_2 & 0 & 0 \\ 0 & & S_c & & 0 \\ 0 & 0 & B_0 & B_1 & B_2 \end{pmatrix}, \quad \widetilde{M} = \begin{pmatrix} \widetilde{A}_0 & \widetilde{A}_1 & \widetilde{A}_2 & 0 & 0 \\ 0 & & \widetilde{S}_c & & 0 \\ 0 & 0 & \widetilde{B}_0 & \widetilde{B}_1 & \widetilde{B}_2 \end{pmatrix},$$

where S_c is a block matrix of interior rows of scaling coefficients $\{s_k\}_{k=0}^{2n+1}$ that were unaltered by the transform. A_0, A_1, A_2, and B_0, B_1, B_2 are $n \times n$ block matrices.

MRA on Intervals

Define

$$V_0 = \text{The linear span of } \{\phi_j(x)\}_{j=0}^{n_f-1}$$

where

$$\phi_j(x) = \begin{cases} \phi_{A,j}(x) & j = 0, \ldots, n-1 \\ \phi(x-j-n) & j = n, \ldots, n_f - n - 1 \\ \phi_{B,j-n_f+n}(x) & j = n_f - n, \ldots, n_f - 1 \end{cases}$$

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The collection is defined by the scaling relation

$$\Phi_0(x) = (\phi_j(x)) = M\Phi_1(2x) \tag{1}$$

(Φ_1 is a column vector (of length $2n_f$) which also contains the functions ϕ_j, but includes more interior functions (more shifts of the scaling function ϕ).)
For instance, in the case of $n = 1$ we would have

$$\phi_A(x) = a_0\phi_A(2x) + a_1\phi(2x) + a_2\phi(2x-1).$$

Biorthogonal MRAs on Intervals

Theorem

There exist unique solutions to (1), $\{\phi_j\}$ *and* $\{\tilde{\phi}_j\}$*, which are compactly supported and satisfy* $\langle \phi_j, \tilde{\phi}_k \rangle = \delta_{j,k}$ *provided that* $|A_0|$, $|\widetilde{A_0}|$, $|B_2|$, *and* $|\widetilde{B_2}| < \sqrt{2}$.

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There exist unique solutions to (1), $\{\phi_j\}$ and $\{\tilde{\phi}_j\}$, which are compactly supported and satisfy $\langle \phi_j, \tilde{\phi}_k \rangle = \delta_{j,k}$ provided that $|A_0|$, $|\widetilde{A}_0|$, $|B_2|$, and $|\widetilde{B_2}| < \sqrt{2}$.

We define W_0 as the linear span of the components of Ψ_0, where

$$\Psi_0(x) = (\psi_j(x)) = N\Phi_1(2x)$$

V_1 is defined as the linear span of the components of $\Phi_1(2x)$.This produces a biresolution analysis over $[0, 2n_f]$ consisting of finite-dimensional subspaces.

Theorem

Given biorthogonal scaling functions $\phi(x)$ and $\tilde{\phi}(x)$ supported in a finite interval $[0, 2n_f]$, we have $V_0 \bigoplus W_0 = V_1$ and $\widetilde{V}_0 \bigoplus \widetilde{W_0} = \widetilde{V}_1$ (as oblique sums) as well as $V_0 \perp \widetilde{W_0}$ and $\widetilde{V}_0 \perp W_0$.

Regularity

Let C^α denote functions which are Hölder continuous of order α.

Theorem

If $\phi \in C^\alpha$ then $\phi_{A,j}(x)$ is C^α for $x > 0$.
If $\phi \in C^\alpha$ and $|A_0| < 2^{-\alpha}$, then $\phi_{A,j}(x)$ is C^α at 0.

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Conjecture

If $\sum_k (-1)^k k^m s_k = \sum_k (-1)^k k^m \tilde{s}_k = 0$ for $m = 0, 1, \ldots, n-1$, then there exist multiresolution analyses of $L^2([0, 2n_f])$ as defined by Theorem 3, such that there exist $c_{k,n}$, $\tilde{c}_{k,n}$ so that for any x in the interval $[0, 2n_f]$,

$$x^m = \sum_k c_{k,n}\phi_k(x) = \sum_k \tilde{c}_{k,n}\tilde{\phi}_k(x), \quad m = 0, 1, \ldots, n.$$

References

Thanks!

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