How To Solve It

A New Aspect of Mathematical Method

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PART I. IN THE CLASSROOM

PURPOSE

1. Helping the student. One of the most important tasks of the teacher is to help his students. This task is not quite easy; it demands time, practice, devotion, and sound principles.

The student should acquire as much experience of independent work as possible. But if he is left alone with his problem without any help or with insufficient help, he may make no progress at all. If the teacher helps too much, nothing is left to the student. The teacher should help, but not too much and not too little, so that the student shall have a *reasonable share of the work.*

If the student is not able to do much, the teacher should leave him at least some illusion of independent work. In order to do so, the teacher should help the student discreetly, *unobtrusively.*

The best is, however, to help the student naturally. The teacher should put himself in the student's place, he should see the student's case, he should try to understand what is going on in the student's mind, and ask a question or indicate a step that *could have occurred to the student himself.*

2. Questions, recommendations, mental operations. Trying to help the student effectively but unobtrusively and naturally, the teacher is led to ask the same questions and to indicate the same steps again and again. Thus, in countless problems, we have to ask the question: *What* *is the unknown?* We may vary the words, and ask the same thing in many different ways: What is required? What do you want to find? What are you supposed to seek? The aim of these questions is to focus the student's attention upon the unknown. Sometimes, we obtain the same effect more naturally with a suggestion: *Look at the unknown!* Question and suggestion aim at the same effect; they tend to provoke the same mental operation.

It seemed to the author that it might be worth while to collect and to group questions and suggestions which are typically helpful in discussing problems with students. The list we study contains questions and suggestions of this sort, carefully chosen and arranged; they are equally useful to the problem-solver who works by himself. If the reader is sufficiently acquainted with the list and can see, behind the suggestion, the action suggested, he may realize that the list enumerates, indirectly, *mental operations typically useful for the solution of problems.* These operations are listed in the order in which they are most likely to occur.

3. **Generality** is an important characteristic of the questions and suggestions contained in our list. Take the questions: *What is the unknown? What are the data? What is the condition?* These questions are generally applicable, we can ask them with good effect dealing with all sorts of problems. Their use is not restricted to any subject-matter. Our problem may be algebraic or geometric, mathematical or nonmathematical, theoretical or practical, a serious problem or a mere puzzle; it makes no difference, the questions make sense and might help us to solve the problem.

There is a **restriction,** in fact, but it has nothing to do with the subject-matter. Certain questions and suggestions of the list are applicable to "problems to find" only, not to "problems to prove." If we have a problem of the latter kind we must use different questions; see PROBLEMS TO FIND, PROBLEMS TO PROVE.

4. **Common sense.** The questions and suggestions of our list are general, but, except for their generality, they are natural, simple, obvious, and proceed from plain common sense. Take the suggestion: *Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.* This suggestion advises you to do what you would do anyhow, without any advice, if you were seriously concerned with your problem. Are you hungry? You wish to obtain food and you think of familiar ways of obtaining food. Have you a problem of geometric construction? You wish to construct a triangle and you think of familiar ways of constructing a triangle. Have you a problem of any kind? You wish to find a certain unknown, and you think of familiar ways of finding such an unknown, or some similar unknown. If you do so you follow exactly the suggestion we quoted from our list. And you are on the right track, too; the suggestion is a good one, it suggests to you a procedure which is very frequently successful.

All the questions and suggestions of our list are natural, simple, obvious, just plain common sense; but they state plain common sense in general terms. They suggest a certain conduct which comes naturally to any person who is seriously concerned with his problem and has some common sense. But the person who behaves the right way usually does not care to express his behavior in clear words and, possibly, he cannot express it so; our list tries to express it so.

5. **Teacher and student. Imitation and practice.** There are two aims which the teacher may have in view when addressing to his students a question or a suggestion of the list: First, to help the student to solve the problem

at hand. Second, to develop the student's ability so that he may solve future problems by himself.

Experience shows that the questions and suggestions of our list, appropriately used, very frequently help the student. They have two common characteristics, common sense and generality. As they proceed from plain common sense they very often come naturally; they could have occurred to the student himself. As they are general, they help unobtrusively; they just indicate a general direction and leave plenty for the student to do.

But the two aims we mentioned before are closely connected; if the student succeeds in solving the problem at hand, he adds a little to his ability to solve problems. Then, we should not forget that our questions are general, applicable in many cases. If the same question is repeatedly helpful, the student will scarcely fail to notice it and he will be induced to ask the question by himself in a similar situation. Asking the question repeatedly, he may succeed once in eliciting the right idea. By such a success, he discovers the right way of using the question, and then he has really assimilated it.

The student may absorb a few questions of our list so well that he is finally able to put to himself the right question in the right moment and to perform the corresponding mental operation naturally and vigorously. Such a student has certainly derived the greatest possible profit from our list. What can the teacher do in order to obtain this best possible result?

Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice. Trying to swim, you imitate what other people do with their hands and feet to keep their heads above water, and, finally, you learn to swim by practicing swimming. Trying to solve problems, you have to observe and to imitate what other people do when solving problems and, finally, you learn to do problems by doing them.

The teacher who wishes to develop his students' ability to do problems must instill some interest for problems into their minds and give them plenty of opportunity for imitation and practice. If the teacher wishes to develop in his students the mental operations which correspond to the questions and suggestions of our list, he puts these questions and suggestions to the students as often as he can do so naturally. Moreover, when the teacher solves a problem before the class, he should dramatize his ideas a little and he should put to himself the same questions which he uses when helping the students. Thanks to such guidance, the student will eventually discover the right use of these questions and suggestions, and doing so he will acquire something that is more important than the knowledge of any particular mathematical fact.

MAIN DIVISIONS, MAIN QUESTIONS

6. Four phases. Trying to find the solution, we may repeatedly change our point of view, our way of looking at the problem. We have to shift our position again and again. Our conception of the problem is likely to be rather incomplete when we start the work; our outlook is different when we have made some progress; it is again different when we have almost obtained the solution.

In order to group conveniently the questions and suggestions of our list, we shall distinguish four phases of the work. First, we have to *understand* the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a *plan*. Third, we *carry out* our plan. Fourth, we *look back* at the completed solution, we review and discuss it.

Each of these phases has its importance. It may happen that a student hits upon an exceptionally bright idea and jumping all preparations blurts out with the solution. Such lucky ideas, of course, are most desirable, but something very undesirable and unfortunate may result if the student leaves out any of the four phases without having a good idea. The worst may happen if the student embarks upon computations or constructions without having *understood* the problem. It is generally useless to carry out details without having seen the main connection, or having made a sort of *plan.* Many mistakes can be avoided if, carrying out his plan, the student *checks each step.* Some of the best effects may be lost if the student fails to reexamine and to *reconsider* the completed solution.

7. Understanding the problem. It is foolish to answer a question that you do not understand. It is sad to work for an end that you do not desire. Such foolish and sad things often happen, in and out of school, but the teacher should try to prevent them from happening in his class. The student should understand the problem. But he should not only understand it, he should also desire its solution. If the student is lacking in understanding or in interest, it is not always his fault; the problem should be well chosen, not too difficult and not too easy, natural and interesting, and some time should be allowed for natural and interesting presentation.

First of all, the verbal statement of the problem must be understood. The teacher can check this, up to a certain extent; he asks the student to repeat the statement, and the student should be able to state the problem fluently. The student should also be able to point out the principal parts of the problem, the unknown, the

data, the condition. Hence, the teacher can seldom afford to miss the questions: *What is the unknown? What are the data? What is the condition?*

The student should consider the principal parts of the problem attentively, repeatedly, and from various sides. If there is a figure connected with the problem he should *draw a figure* and point out on it the unknown and the data. If it is necessary to give names to these objects he should *introduce suitable notation*; devoting some attention to the appropriate choice of signs, he is obliged to consider the objects for which the signs have to be chosen. There is another question which may be useful in this preparatory stage provided that we do not expect a definitive answer but just a provisional answer, a guess: *Is it possible to satisfy the condition?*

(In the exposition of Part II [p. 33] "Understanding the problem" is subdivided into two stages: "Getting acquainted" and "Working for better understanding.")

8. Example. Let us illustrate some of the points explained in the foregoing section. We take the following simple problem: *Find the diagonal of a rectangular parallelepiped of which the length, the width, and the height are known.*

In order to discuss this problem profitably, the students must be familiar with the theorem of Pythagoras, and with some of its applications in plane geometry, but they may have very little systematic knowledge in solid geometry. The teacher may rely here upon the student's unsophisticated familiarity with spatial relations.

The teacher can make the problem interesting by making it concrete. The classroom is a rectangular parallelepiped whose dimensions could be measured, and can be estimated; the students have to find, too "measure indirectly," the diagonal of the classroom. The teacher points out the length, the width, and the height of the classroom, indicates the diagonal with a gesture, and enlivens his figure, drawn on the blackboard, by referring repeatedly to the classroom.

The dialogue between the teacher and the students may start as follows:

"What is the unknown?"

"The length of the diagonal of a parallelepiped."

"What are the data?"

"The length, the width, and the height of the parallelepiped."

"Introduce suitable notation. Which letter should denote the unknown?"

"x."

"Which letters would you choose for the length, the width, and the height?"

"a, b, c."

"What is the condition, linking a, b, c, and x?"

"x is the diagonal of the parallelepiped of which a, b, and c are the length, the width, and the height."

"Is it a reasonable problem? I mean, *is the condition sufficient to determine the unknown?"*

"Yes, it is. If we know a, b, c, we know the parallelepiped. If the parallelepiped is determined, the diagonal is determined."

9. Devising a plan. We have a plan when we know, or know at least in outline, which calculations, computations, or constructions we have to perform in order to obtain the unknown. The way from understanding the problem to conceiving a plan may be long and tortuous. In fact, the main achievement in the solution of a problem is to conceive the idea of a plan. This idea may emerge gradually. Or, after apparently unsuccessful trials and a period of hesitation, it may occur suddenly, in a flash, as a "bright idea." The best that the teacher can do for the student is to procure for him, by unobtrusive

help, a bright idea. The questions and suggestions we are going to discuss tend to provoke such an idea.

In order to be able to see the student's position, the teacher should think of his own experience, of his difficulties and successes in solving problems.

We know, of course, that it is hard to have a good idea if we have little knowledge of the subject, and impossible to have it if we have no knowledge. Good ideas are based on past experience and formerly acquired knowledge. Mere remembering is not enough for a good idea, but we cannot have any good idea without recollecting some pertinent facts; materials alone are not enough for constructing a house but we cannot construct a house without collecting the necessary materials. The materials necessary for solving a mathematical problem are certain relevant items of our formerly acquired mathematical knowledge, as formerly solved problems, or formerly proved theorems. Thus, it is often appropriate to start the work with the question: *Do you know a related problem?*

The difficulty is that there are usually too many problems which are somewhat related to our present problem, that is, have some point in common with it. How can we choose the one, or the few, which are really useful? There is a suggestion that puts our finger on an essential common point: *Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.*

If we succeed in recalling a formerly solved problem which is closely related to our present problem, we are lucky. We should try to deserve such luck; we may deserve it by exploiting it. *Here is a problem related to yours and solved before. Could you use it?*

The foregoing questions, well understood and seriously considered, very often help to start the right train of ideas; but they cannot help always, they cannot work magic. If they do not work, we must look around for some other appropriate point of contact, and explore the various aspects of our problem; we have to vary, to transform, to modify the problem. *Could you restate the problem?* Some of the questions of our list hint specific means to vary the problem, as generalization, specialization, use of analogy, dropping a part of the condition, and so on; the details are important but we cannot go into them now. Variation of the problem may lead to some appropriate auxiliary problem: *If you cannot solve the proposed problem try to solve first some related problem.*

Trying to apply various known problems or theorems, considering various modifications, experimenting with various auxiliary problems, we may stray so far from our original problem that we are in danger of losing it altogether. Yet there is a good question that may bring us back to it: *Did you use all the data? Did you use the whole condition?*

10. Example. We return to the example considered in section 8. As we left it, the students just succeeded in understanding the problem and showed some mild interest in it. They could now have some ideas of their own, some initiative. If the teacher, having watched sharply, cannot detect any sign of such initiative he has to resume carefully his dialogue with the students. He must be prepared to repeat with some modification the questions which the students do not answer. He must be prepared to meet often with the disconcerting silence of the students (which will be indicated by dots . . . . .).

"Do you know a related problem?"

. . . . .

"Look at the unknown! Do you know a problem having the same unknown?"

. . . . .

"Well, *what is the unknown?*"

"The diagonal of a parallelepiped."

"Do you know any *problem with the same unknown?*"

"No. We have not had any problem yet about the diagonal of a parallelepiped."

"Do you know any *problem with a similar unknown?*"

. . . . .

"You see, the diagonal is a segment, the segment of a straight line. Did you never solve a problem whose unknown was the length of a line?"

"Of course, we have solved such problems. For instance, to find a side of a right triangle."

"Good! *Here is a problem related to yours and solved before. Could you use it?*"

. . . . .

"You were lucky enough to remember a problem which is related to your present one and which you solved

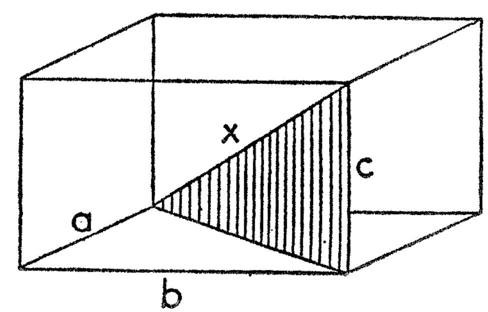

FIG. 1

before. Would you like to use it? *Could you introduce some auxiliary element in order to make its use possible?*"

. . . . .

"Look here, the problem you remembered is about a triangle. Have you any triangle in your figure?"

Let us hope that the last hint was explicit enough to provoke the idea of the solution which is to introduce a right triangle, (emphasized in Fig. 1) of which the required diagonal is the hypotenuse. Yet the teacher should be prepared for the case that even this fairly explicit hint is insufficient to shake the torpor of the students; and so he should be prepared to use a whole gamut of more and more explicit hints.

"Would you like to have a triangle in the figure?"

"What sort of triangle would you like to have in the figure?"

"You cannot find yet the diagonal; but you said that you could find the side of a triangle. Now, what will you do?"

"Could you find the diagonal, if it were a side of a triangle?"

When, eventually, with more or less help, the students succeed in introducing the decisive auxiliary element, the right triangle emphasized in Fig. 1, the teacher should convince himself that the students see sufficiently far ahead before encouraging them to go into actual calculations.

"I think that it was a good idea to draw that triangle. You have now a triangle; but have you the unknown?"

"The unknown is the hypotenuse of the triangle; we can calculate it by the theorem of Pythagoras."

"You can, if both legs are known; but are they?"

"One leg is given, it is c. And the other, I think, is not difficult to find. Yes, the other leg is the hypotenuse of another right triangle."

"Very good! Now I see that you have a plan."

11. Carrying out the plan. To devise a plan, to conceive the idea of the solution is not easy. It takes so much to succeed; formerly acquired knowledge, good mental habits, concentration upon the purpose, and one more thing: good luck. To carry out the plan is much easier; what we need is mainly patience.

The plan gives a general outline; we have to convince

ourselves that the details fit into the outline, and so we have to examine the details one after the other, patiently, till everything is perfectly clear, and no obscure corner remains in which an error could be hidden.

If the student has really conceived a plan, the teacher has now a relatively peaceful time. The main danger is that the student forgets his plan. This may easily happen if the student received his plan from outside, and accepted it on the authority of the teacher; but if he worked for it himself, even with some help, and conceived the final idea with satisfaction, he will not lose this idea easily. Yet the teacher must insist that the student should *check each step.*

We may convince ourselves of the correctness of a step in our reasoning either "intuitively" or "formally." We may concentrate upon the point in question till we see it so clearly and distinctly that we have no doubt that the step is correct; or we may derive the point in question according to formal rules. (The difference between "insight" and "formal proof" is clear enough in many important cases; we may leave further discussion to philosophers.)

The main point is that the student should be honestly convinced of the correctness of each step. In certain cases, the teacher may emphasize the difference between "seeing" and "proving": *Can you see clearly that the step is correct?* But can you also *prove that the step is correct?*

12. Example. Let us resume our work at the point where we left it at the end of section **10**. The student, at last, has got the idea of the solution. He sees the right triangle of which the unknown x is the hypotenuse and the given height c is one of the legs; the other leg is the diagonal of a face. The student must, possibly, be urged to introduce suitable notation. He should choose y to denote that other leg, the diagonal of the face whose sides are a and b. Thus, he may see more clearly the idea of the solution which is to introduce an auxiliary problem whose unknown is y. Finally, working at one right triangle after the other, he may obtain (see Fig. 1)

$$x^2 = y^2 + c^2$$
$$y^2 = a^2 + b^2$$

and hence, eliminating the auxiliary unknown y,

$$x^2 = a^2 + b^2 + c^2$$
$$x = \sqrt{a^2 + b^2 + c^2}.$$

The teacher has no reason to interrupt the student if he carries out these details correctly except, possibly, to warn him that he should *check each step*. Thus, the teacher may ask:

"Can you *see clearly* that the triangle with sides x, y, c is a right triangle?"

To this question the student may answer honestly "Yes" but he could be much embarrassed if the teacher, not satisfied with the intuitive conviction of the student, should go on asking:

"But can you *prove* that this triangle is a right triangle?"

Thus, the teacher should rather suppress this question unless the class has had a good initiation in solid geometry. Even in the latter case, there is some danger that the answer to an incidental question may become the main difficulty for the majority of the students.

13. Looking back. Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. By looking back at the completed solution, by reconsidering and reexamining the result and the path that led to it, they could consolidate their knowledge and develop their ability to solve problems. A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution.

The student has now carried through his plan. He has written down the solution, checking each step. Thus, he should have good reasons to believe that his solution is correct. Nevertheless, errors are always possible, especially if the argument is long and involved. Hence, verifications are desirable. Especially, if there is some rapid and intuitive procedure to test either the result or the argument, it should not be overlooked. *Can you check the result? Can you check the argument?*

In order to convince ourselves of the presence or of the quality of an object, we like to see and to touch it. And as we prefer perception through two different senses, so we prefer conviction by two different proofs: *Can you derive the result differently?* We prefer, of course, a short and intuitive argument to a long and heavy one: *Can you see it at a glance?*

One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution. The students will find looking back at the solution really interesting if they have made an honest effort, and have the consciousness of having done well. Then they are eager to see what else they could accomplish with that effort, and how they could do equally well another time. The teacher should encourage the students to imagine cases in which they could utilize again the procedure used, or apply the result obtained. *Can you use the result, or the method, for some other problem?*

14. Example. In section 12, the students finally obtained the solution: If the three edges of a rectangular parallelogram, issued from the same corner, are a, b, c, the diagonal is

$$\sqrt{a^2 + b^2 + c^2}.$$

Can you check the result? The teacher cannot expect a good answer to this question from inexperienced students. The students, however, should acquire fairly early the experience that problems "in letters" have a great advantage over purely numerical problems; if the problem is given "in letters" its result is accessible to several tests to which a problem "in numbers" is not susceptible at all. Our example, although fairly simple, is sufficient to show this. The teacher can ask several questions about the result which the students may readily answer with "Yes"; but an answer "No" would show a serious flaw in the result.

"*Did you use all the data?* Do all the data a, b, c appear in your formula for the diagonal?"

"Length, width, and height play the same role in our question; our problem is symmetric with respect to a, b, c. Is the expression you obtained for the diagonal symmetric in a, b, c? Does it remain unchanged when a, b, c are interchanged?"

"Our problem is a problem of solid geometry: to find the diagonal of a parallelepiped with given dimensions a, b, c. Our problem is analogous to a problem of plane geometry: to find the diagonal of a rectangle with given dimensions a, b. Is the result of our 'solid' problem analogous to the result of the 'plane' problem?"

"If the height c decreases, and finally vanishes, the

parallelepiped becomes a parallelogram. If you put $c = 0$ in your formula, do you obtain the correct formula for the diagonal of the rectangular parallelogram?"

"If the height c increases, the diagonal increases. Does your formula show this?"

"If all three measures a, b, c of the parallelepiped increase in the same proportion, the diagonal also increases in the same proportion. If, in your formula, you substitute $12a$, $12b$, $12c$ for a, b, c respectively, the expression of the diagonal, owing to this substitution, should also be multiplied by 12. Is that so?"

"If a, b, c are measured in feet, your formula gives the diagonal measured in feet too; but if you change all measures into inches, the formula should remain correct. Is that so?"

(The two last questions are essentially equivalent; see TEST BY DIMENSION.)

These questions have several good effects. First, an intelligent student cannot help being impressed by the fact that the formula passes so many tests. He was convinced before that the formula is correct because he derived it carefully. But now he is more convinced, and his gain in confidence comes from a different source; it is due to a sort of "experimental evidence." Then, thanks to the foregoing questions, the details of the formula acquire new significance, and are linked up with various facts. The formula has therefore a better chance of being remembered, the knowledge of the student is consolidated. Finally, these questions can be easily transferred to similar problems. After some experience with similar problems, an intelligent student may perceive the underlying general ideas: use of all relevant data, variation of the data, symmetry, analogy. If he gets into the habit of directing his attention to such points, his ability to solve problems may definitely profit.

Can you check the argument? To recheck the argument step by step may be necessary in difficult and important cases. Usually, it is enough to pick out "touchy" points for rechecking. In our case, it may be advisable to discuss retrospectively the question which was less advisable to discuss as the solution was not yet attained: Can you *prove* that the triangle with sides x, y, c is a right triangle? (See the end of section **12.**)

Can you use the result or the method for some other problem? With a little encouragement, and after one or two examples, the students easily find applications which consist essentially in giving some *concrete interpretation* to the abstract mathematical elements of the problem. The teacher himself used such a concrete interpretation as he took the room in which the discussion takes place for the parallelepiped of the problem. A dull student may propose, as application, to calculate the diagonal of the cafeteria instead of the diagonal of the classroom. If the students do not volunteer more imaginative remarks, the teacher himself may put a slightly different problem, for instance: "Being given the length, the width, and the height of a rectangular parallelepiped, find the distance of the center from one of the corners."

The students may use the *result* of the problem they just solved, observing that the distance required is one half of the diagonal they just calculated. Or they may use the *method,* introducing suitable right triangles (the latter alternative is less obvious and somewhat more clumsy in the present case).

After this application, the teacher may discuss the configuration of the four diagonals of the parallelepiped, and the six pyramids of which the six faces are the bases, the center the common vertex, and the semidiagonals the lateral edges. When the geometric imagination of the students is sufficiently enlivened, the teacher should come back to his question: *Can you use the result, or the method, for some other problem?* Now there is a better chance that the students may find some more interesting concrete interpretation, for instance, the following:

"In the center of the flat rectangular top of a building which is 21 yards long and 16 yards wide, a flagpole is to be erected, 8 yards high. To support the pole, we need four equal cables. The cables should start from the same point, 2 yards under the top of the pole, and end at the four corners of the top of the building. How long is each cable?"

The students may use the *method* of the problem they solved in detail introducing a right triangle in a vertical plane, and another one in a horizontal plane. Or they may use the *result,* imagining a rectangular parallelepiped of which the diagonal, x, is one of the four cables and the edges are

$$a = 10.5 \qquad b = 8 \qquad c = 6.$$

By straightforward application of the formula, $x = 14.5$.

For more examples, see CAN YOU USE THE RESULT?

15. Various approaches. Let us still retain, for a while, the problem we considered in the foregoing sections **8, 10, 12, 14.** The main work, the discovery of the plan, was described in section **10.** Let us observe that the teacher could have proceeded differently. Starting from the same point as in section **10,** he could have followed a somewhat different line, asking the following questions:

"Do you know any related problem?"

"Do you know an *analogous* problem?"

"You see, the proposed problem is a problem of solid geometry. Could you think of a simpler analogous problem of plane geometry?"

"You see, the proposed problem is about a figure in space, it is concerned with the diagonal of a rectangular parallelepiped. What might be an analogous problem about a figure in the plane? It should be concerned with —the diagonal—of—a rectangular—"

"Parallelogram."

The students, even if they are very slow and indifferent, and were not able to guess anything before, are obliged finally to contribute at least a minute part of the idea. Besides, if the students are so slow, the teacher should not take up the present problem about the parallelepiped without having discussed before, in order to prepare the students, the analogous problem about the parallelogram. Then, he can go on now as follows:

"Here is a problem related to yours and solved before. Can you use it?"

"Should you introduce some auxiliary element in order to make its use possible?"

Eventually, the teacher may succeed in suggesting to the students the desirable idea. It consists in conceiving the diagonal of the given parallelepiped as the diagonal of a suitable parallelogram which must be introduced into the figure (as intersection of the parallelepiped with a plane passing through two opposite edges). The idea is essentially the same as before (section **10**) but the approach is different. In section **10**, the contact with the available knowledge of the students was established through the unknown; a formerly solved problem was recollected because its unknown was the same as that of the proposed problem. In the present section analogy provides the contact with the idea of the solution.

16. The teacher's method of questioning shown in the foregoing sections **8, 10, 12, 14, 15** is essentially this: Begin with a general question or suggestion of our list, and, if necessary, come down gradually to more specific and concrete questions or suggestions till you reach one which elicits a response in the student's mind. If you

have to help the student exploit his idea, start again, if possible, from a general question or suggestion contained in the list, and return again to some more special one if necessary; and so on.

Of course, our list is just a first list of this kind; it seems to be sufficient for the majority of simple cases, but there is no doubt that it could be perfected. It is important, however, that the suggestions from which we start should be simple, natural, and general, and that their list should be short.

The suggestions must be simple and natural because otherwise they cannot be *unobtrusive.*

The suggestions must be general, applicable not only to the present problem but to problems of all sorts, if they are to help develop the *ability* of the student and not just a special technique.

The list must be short in order that the questions may be often repeated, unartificially, and under varying circumstances; thus, there is a chance that they will be eventually assimilated by the student and will contribute to the development of a *mental habit.*

It is necessary to come down gradually to specific suggestions, in order that the student may have as great a *share of the work* as possible.

This method of questioning is not a rigid one; fortunately so, because, in these matters, any rigid, mechanical, pedantical procedure is necessarily bad. Our method admits a certain elasticity and variation, it admits various approaches (section **15**), it can be and should be so applied that questions asked by the teacher *could have occurred to the student himself.*

If a reader wishes to try the method here proposed in his class he should, of course, proceed with caution. He should study carefully the example introduced in section **8**, and the following examples in sections **18, 19, 20**. He should prepare carefully the examples which he intends to discuss, considering also various approaches. He should start with a few trials and find out gradually how he can manage the method, how the students take it, and how much time it takes.

17. Good questions and bad questions. If the method of questioning formulated in the foregoing section is well understood it helps to judge, by comparison, the quality of certain suggestions which may be offered with the intention of helping the students.

Let us go back to the situation as it presented itself at the beginning of section **10** when the question was asked: *Do you know a related problem?* Instead of this, with the best intention to help the students, the question may be offered: *Could you apply the theorem of Pythagoras?*

The intention may be the best, but the question is about the worst. We must realize in what situation it was offered; then we shall see that there is a long sequence of objections against that sort of "help."

(1) If the student is near to the solution, he may understand the suggestion implied by the question; but if he is not, he quite possibly will not see at all the point at which the question is driving. Thus the question fails to help where help is most needed.

(2) If the suggestion is understood, it gives the whole secret away, very little remains for the student to do.

(3) The suggestion is of too special a nature. Even if the student can make use of it in solving the present problem, nothing is learned for future problems. The question is not instructive.

(4) Even if he understands the suggestion, the student can scarcely understand how the teacher came to the idea of putting such a question. And how could he, the student, find such a question by himself? It appears as an unnatural surprise, as a rabbit pulled out of a hat; it is really not instructive.

None of these objections can be raised against the procedure described in section **10**, or against that in section **15**.

MORE EXAMPLES

18. A problem of construction. *Inscribe a square in a given triangle. Two vertices of the square should be on the base of the triangle, the two other vertices of the square on the two other sides of the triangle, one on each.*

"What is the unknown?"

"A square."

"What are the data?"

"A triangle is given, nothing else."

"What is the condition?"

"The four corners of the square should be on the perimeter of the triangle, two corners on the base, one corner on each of the other two sides."

"Is it possible to satisfy the condition?"

"I think so. I am not so sure."

"You do not seem to find the problem too easy. *If you cannot solve the proposed problem, try to solve first some related problem.* Could you satisfy a *part of the condition?*"

"What do you mean by a part of the condition?"

"You see, the condition is concerned with all the vertices of the square. How many vertices are there?"

"Four."

"A part of the condition would be concerned with less than four vertices. *Keep only a part of the condition, drop the other part.* What part of the condition is easy to satisfy?"

"It is easy to draw a square with two vertices on the perimeter of the triangle—or even one with three vertices on the perimeter!"

"Draw a figure!"

The student draws Fig. 2.

"You *kept only a part of the condition,* and you *dropped the other part. How far is the unknown now determined?*"

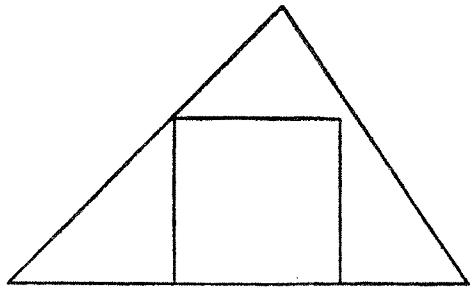

FIG. 2

"The square is not determined if it has only three vertices on the perimeter of the triangle."

"Good! *Draw a figure.*"

The student draws Fig. 3.

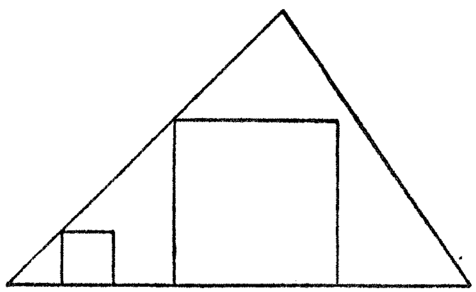

FIG. 3

"The square, as you said, is not determined by the *part of the condition you kept. How can it vary?*"

. . . . .

"Three corners of your square are on the perimeter of the triangle but the fourth corner is not yet there where it should be. Your square, as you said, is undetermined,

it can vary; the same is true of its fourth corner. *How can it vary?*"

. . . . .

"Try it experimentally, if you wish. Draw more squares with three corners on the perimeter in the same way as the two squares already in the figure. Draw small squares and large squares. What seems to be the locus of the fourth corner? *How can it vary?*"

The teacher brought the student very near to the idea of the solution. If the student is able to guess that the locus of the fourth corner is a straight line, he has got it.

19. A problem to prove. *Two angles are in different planes but each side of one is parallel to the corresponding side of the other, and has also the same direction. Prove that such angles are equal.*

What we have to prove is a fundamental theorem of solid geometry. The problem may be proposed to students who are familiar with plane geometry and acquainted with those few facts of solid geometry which prepare the present theorem in Euclid's Elements. (The theorem that we have stated and are going to prove is the proposition 10 of Book XI of Euclid.) Not only questions and suggestions quoted from our list are printed in italics but also others which correspond to them as "problems to prove" correspond to "problems to find." (The correspondence is worked out systematically in PROBLEMS TO FIND, PROBLEMS TO PROVE 5, 6.)

"What is the hypothesis?"

"Two angles are in different planes. Each side of one is parallel to the corresponding side of the other, and has also the same direction.

"What is the conclusion?"

"The angles are equal."

"Draw a figure. Introduce suitable notation."

The student draws the lines of Fig. 4 and chooses, helped more or less by the teacher, the letters as in Fig. 4.

"*What is the hypothesis?* Say it, please, using your notation."

"A, B, C are not in the same plane as A', B', C'. And $AB \parallel A'B'$, $AC \parallel A'C'$. Also AB has the same direction as $A'B'$, and AC the same as $A'C'$."

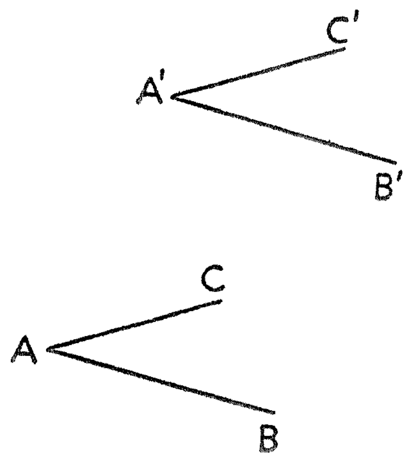

FIG. 4

"*What is the conclusion?*"

"$\angle BAC = \angle B'A'C'$."

"*Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.*"

"If two triangles are congruent, the corresponding angles are equal."

"Very good! Now *here is a theorem related to yours and proved before. Could you use it?*"

"I think so but I do not see yet quite how."

"*Should you introduce some auxiliary element in order to make its use possible?*"

. . . . .

"Well, the theorem which you quoted so well is about

triangles, about a pair of congruent triangles. Have you any triangles in your figure?"

"No. But I could introduce some. Let me join B to C, and B' to C'. Then there are two triangles, $\triangle ABC$, $\triangle A'B'C'$."

"Well done. But what are these triangles good for?"

"To prove the conclusion, $\angle BAC = \angle B'A'C'$."

"Good! If you wish to prove this, what kind of triangles do you need?"

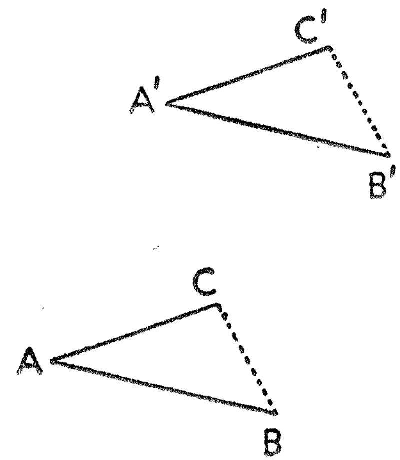

FIG. 5

"Congruent triangles. Yes, of course, I may choose B, C, B', C' so that

$$AB = A'B', AC = A'C'."$$

"Very good! Now, what do you wish to prove?"

"I wish to prove that the triangles are congruent,

$$\triangle ABC = \triangle A'B'C'.$$

If I could prove this, the conclusion $\angle BAC = \angle B'A'C'$ would follow immediately."

"Fine! You have a new aim, you aim at a new conclusion. *Look at the conclusion! And try to think of a* *familiar theorem having the same or a similar conclusion."*

"Two triangles are congruent if—if the three sides of the one are equal respectively to the three sides of the other."

"Well done. You could have chosen a worse one. Now *here is a theorem related to yours and proved before. Could you use it?"*

"I could use it if I knew that $BC = B'C'$."

"That is right! Thus, what is your aim?"

"To prove that $BC = B'C'$."

"Try to think of a familiar theorem having the same or a similar conclusion."

"Yes, I know a theorem finishing: '. . . then the two lines are equal.' But it does not fit in."

"Should you introduce some auxiliary element in order to make its use possible?"

. . . . .

"You see, how could you prove $BC = B'C'$ when there is no connection in the figure between BC and $B'C'$?"

. . . . .

"Did you use the hypothesis? What is the hypothesis?"

"We suppose that $AB \parallel A'B'$, $AC \parallel A'C'$. Yes, of course, I must use that."

"Did you use the whole hypothesis? You say that $AB \parallel A'B'$. Is that all that you know about these lines?"

"No; AB is also equal to $A'B'$, by construction. They are parallel and equal to each other. And so are AC and $A'C'$."

"Two parallel lines of equal length—it is an interesting configuration. *Have you seen it before?"*

"Of course! Yes! Parallelogram! Let me join A to A', B to B', and C to C'."

"The idea is not so bad. How many parallelograms have you now in your figure?"

"Two. No, three. No, two. I mean, there are two of

which you can prove immediately that they are parallelograms. There is a third which seems to be a parallelogram; I hope I can prove that it is one. And then the proof will be finished!"

We could have gathered from his foregoing answers that the student is intelligent. But after this last remark of his, there is no doubt.

This student is able to guess a mathematical result and to distinguish clearly between proof and guess. He knows also that guesses can be more or less plausible. Really, he did profit something from his mathematics classes; he has some real experience in solving problems, he can conceive and exploit a good idea.

20. A rate problem. *Water is flowing into a conical vessel at the rate* r. *The vessel has the shape of a right circular cone, with horizontal base, the vertex pointing downwards; the radius of the base is* a, *the altitude of the*

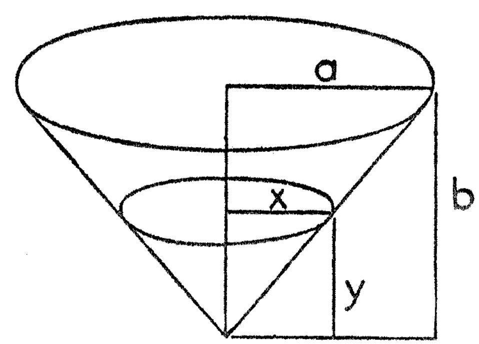

FIG. 6

cone b. *Find the rate at which the surface is rising when the depth of the water is* y. *Finally, obtain the numerical value of the unknown supposing that* a = 4 *ft.,* b = 3 *ft.,* r = 2 *cu. ft. per minute, and* y = 1 *ft.*

The students are supposed to know the simplest rules of differentiation and the notion of "rate of change."

"What are the data?"

"The radius of the base of the cone $a = 4$ ft., the altitude of the cone $b = 3$ ft., the rate at which the water is flowing into the vessel $r = 2$ cu. ft. per minute, and the depth of the water at a certain moment, $y = 1$ ft."

"Correct. The statement of the problem seems to suggest that you should disregard, provisionally, the numerical values, work with the letters, express the unknown in terms of a, b, r, y and only finally, after having obtained the expression of the unknown in letters, substitute the numerical values. I would follow this suggestion. Now, *what is the unknown?*"

"The rate at which the surface is rising when the depth of the water is y."

"What is that? Could you say it in other terms?"

"The rate at which the depth of the water is increasing."

"What is that? *Could you restate it still differently?*"

"The rate of change of the depth of the water."

"That is right, the rate of change of y. But what is the rate of change? *Go back to the definition.*"

"The derivative is the rate of change of a function."

"Correct. Now, is y a function? As we said before, we disregard the numerical value of y. Can you imagine that y changes?"

"Yes, y, the depth of the water, increases as the time goes by."

"Thus, y is a function of what?"

"Of the time t."

"Good. *Introduce suitable notation.* How would you write the 'rate of change of y' in mathematical symbols?"

"$\dfrac{dy}{dt}$"

"Good. Thus, this is your unknown. You have to express it in terms of a, b, r, y. By the way, one of these data is a 'rate.' Which one?"

"r is the rate at which water is flowing into the vessel."

"What is that? Could you say it in other terms?"

"r is the rate of change of the volume of the water in the vessel."

"What is that? *Could you restate it still differently?* How would you write it in *suitable notation?*"

"$r = \dfrac{dV}{dt}$."

"What is V?"

"The volume of the water in the vessel at the time t."

"Good. Thus, you have to express $\dfrac{dy}{dt}$ in terms of a, b, $\dfrac{dV}{dt}$, y. How will you do it?"

. . . . .

"*If you cannot solve the proposed problem try to solve first some related problem.* If you do not see yet the connection between $\dfrac{dy}{dt}$ and the data, try to bring in some simpler connection that could serve as a stepping stone."

. . . . .

"Do you not see that there are other connections? For instance, are y and V independent of each other?"

"No. When y increases, V must increase too."

"Thus, there is a connection. What is the connection?"

"Well, V is the volume of a cone of which the altitude is y. But I do not know yet the radius of the base."

"You may consider it, nevertheless. Call it something, say x."

"$V = \dfrac{\pi x^2 y}{3}$."

"Correct. Now, what about x? Is it independent of y?"

"No. When the depth of the water, y, increases the radius of the free surface, x, increases too."

"Thus, there is a connection. What is the connection?"

"Of course, similar triangles.

$$x : y = a : b."$$

"One more connection, you see. I would not miss profiting from it. Do not forget, you wished to know the connection between V and y."

"I have

$$x = \frac{ay}{b}$$

$$V = \frac{\pi a^2 y^3}{3b^2}."$$

"Very good. This looks like a stepping stone, does it not? But you should not forget your goal. *What is the unknown?*"

"Well, $\frac{dy}{dt}$."

"You have to find a connection between $\frac{dy}{dt}$, $\frac{dV}{dt}$, and other quantities. And here you have one between y, V, and other quantities. What to do?"

"Differentiate! Of course!

$$\frac{dV}{dt} = \frac{\pi a^2 y^2}{b^2} \frac{dy}{dt}.$$

Here it is."

"Fine! And what about the numerical values?"

"If $a = 4$, $b = 3$, $\frac{dV}{dt} = r = 2$, $y = 1$, then

$$2 = \frac{\pi \times 16 \times 1}{9} \frac{dy}{dt}."$$

PART II. HOW TO SOLVE IT
A DIALOGUE

Getting Acquainted

Where should I start? Start from the statement of the problem.

What can I do? Visualize the problem as a whole as clearly and as vividly as you can. Do not concern yourself with details for the moment.

What can I gain by doing so? You should understand the problem, familiarize yourself with it, impress its purpose on your mind. The attention bestowed on the problem may also stimulate your memory and prepare for the recollection of relevant points.

Working for Better Understanding

Where should I start? Start again from the statement of the problem. Start when this statement is so clear to you and so well impressed on your mind that you may lose sight of it for a while without fear of losing it altogether.

What can I do? Isolate the principal parts of your problem. The hypothesis and the conclusion are the principal parts of a "problem to prove"; the unknown, the data, and the conditions are the principal parts of a "problem to find." Go through the principal parts of your problem, consider them one by one, consider them in turn, consider them in various combinations, relating each detail to other details and each to the whole of the problem.