Finite Biorthogonal Transforms and Multiresolution Analyses on Intervals

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Overview

- Background:
  - Scaling Functions
  - Multiresolution Analyses (MRA) and Wavelet Functions
  - The Discrete Wavelet Transform (DWT)
- Motivation: Finite Orthogonal Transforms and MRA on Intervals (Madych, 1997)
- Finite Biorthogonal Transforms and MRA on Intervals

Scaling Functions

Definition

A function $\phi : \mathbb{R} \to \mathbb{C}$ is called a *scaling function* (with *scaling coefficients* $\{s_k\}$) if it satisfies the *refinement equation*:

$$\phi(x) = \sum_k s_k \phi(2x - k)$$

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$$\phi(x) = \sum_k s_k \phi(2x - k)$$

Let

$$m(\xi) = \frac{1}{2} \sum_k s_k e^{-ik\xi}$$

then

$$\hat{\phi}(\xi) = m\left(\frac{\xi}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right)$$

i.e.

$$\hat{\phi}(\xi) = \prod_{k=1}^{\infty} m\left(\frac{\xi}{2^k}\right) \hat{\phi}(0)$$

A scaling function could be interpreted as a fixed point of the operator

$$Tf = \sum_k s_k f(2x - k)$$

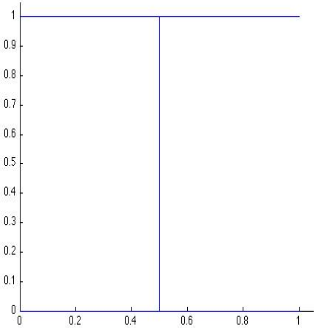
and if ϕ is continuous, it can be defined iteratively by

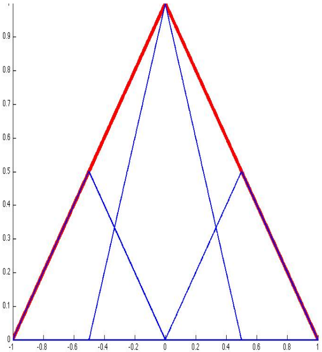
$$\phi^{(n+1)}(x) = \sum_k s_k \phi^{(n)}(2x - k)$$

Examples of Scaling Functions

The characteristic (Haar) function on $[0,1]$ with coefficients $\{s_0, s_1\} = \{1, 1\}$

The hat function on $[-1,1]$ with coefficients $\{s_{-1}, s_0, s_1\} = \{\frac{1}{2}, 1, \frac{1}{2}\}$





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To see this, notice

$$\delta_{0,j} = \langle \phi(x), \phi(x-j) \rangle = \left\langle \sum_k s_k \phi(2x-k), \sum_n s_n \phi(2x-2j-n) \right\rangle$$

$$= \sum_{k,n} s_k s_n \langle \phi(2x-k), \phi(2x-(2j+n)) \rangle$$

$$= \frac{1}{2} \sum_{k,m} s_k s_{m-2j} \langle \phi(t-k), \phi(t-m) \rangle = \frac{1}{2} \sum_k s_k s_{k-2j}$$

MRA

Definition

A *Multiresolution Analysis* (MRA) of $L^2(\mathbb{R})$ is an infinite nested sequence of subspaces of $L^2(\mathbb{R})$

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$$

with the properties

(i) $\bigcup_n V_n$ is dense in $L^2(\mathbb{R})$

(ii) $\bigcap_n V_n = \{0\}$

(iii) $f(x) \in V_n \iff f(2x) \in V_{n+1}$ for all n $\in \mathbb{Z}$

(iv) $f(x) \in V_n \iff f(x - 2^{-n}k) \in V_n$ for all n, k $\in \mathbb{Z}$

(v) $\exists \phi$ such that $\{\phi(x-k) : k \in \mathbb{Z}\}$ forms an orthonormal basis of V_0

$\{\phi_{n,k} = 2^{\frac{n}{2}}\phi(2^n x - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis for V_n.

For ϕ orthogonal, $P_n : L^2 \to V_n$ is given by

$$P_n(f) = \sum_k \langle f, \phi_{n,k} \rangle \, \phi_{n,k}$$

Functions in V_n are said to have *resolution* or *scale* 2^{-n}.
$P_n f$ is an *approximation to f at resolution* 2^{-n}, and $P_n f \to f$ in L^2.

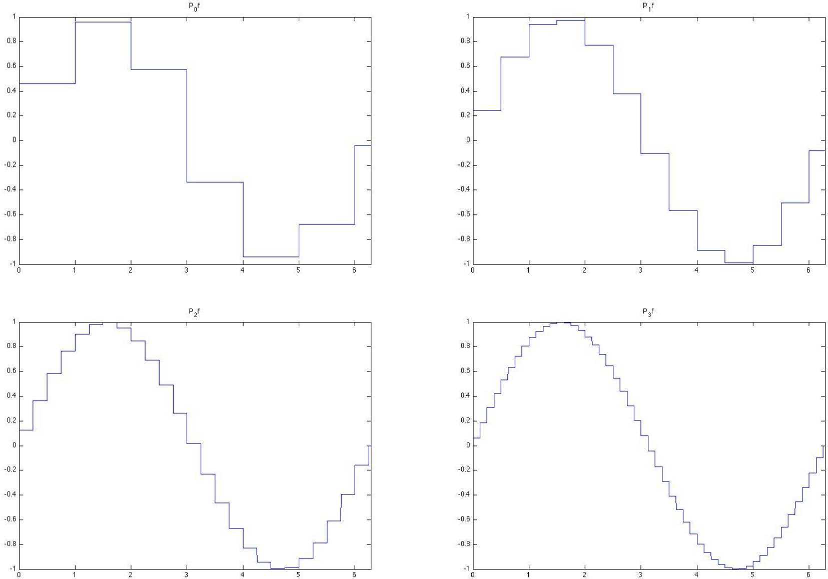
The *fine detail at resolution* 2^{-n} is defined:

$$Q_n f(x) = P_{n+1} f(x) - P_n f(x)$$

The range of Q_n coincides with W_n, the orthogonal complement of V_n within V_{n+1}. We have

$$W_n \perp V_n \qquad \text{and} \qquad V_n \oplus W_n = V_{n+1}$$

Approximations of Sine



Wavelets

The spaces $\{W_n\}$ satisfy conditions similar to that of a MRA. For a MRA with orthogonal scaling function ϕ, the following can be demonstrated:

(i) $\bigoplus_n W_n = L^2$

(ii) $W_k \perp W_n$ if $n \neq k$

(iii) $f(x) \in W_n \iff f(2x) \in W_{n+1}$

(iv) $f(x) \in W_n \iff f(x - 2^{-n}k) \in W_n$

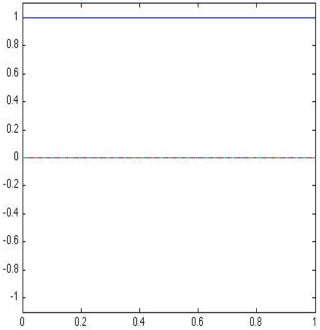
(v) $\exists \psi \in W_0$ such that $\{\psi(x-k)\}$forms an orthonormal basis of W_0.
($\implies \{\psi_{n,k} = 2^{\frac{n}{2}}\psi(2^n x - k) : k \in \mathbb{Z}\}$ forms a basis for W_n.)

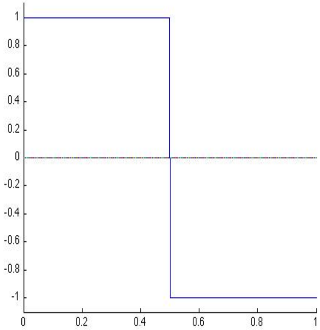
(vi) $\psi(x) = \sum_k w_k \phi(2x-k)$ for some coefficients w_k.
It can be shown that $w_k = (-1)^k s_{N-k}$ for any odd N.

ψ is called the *wavelet function*.

Haar Scaling and Wavelet Functions

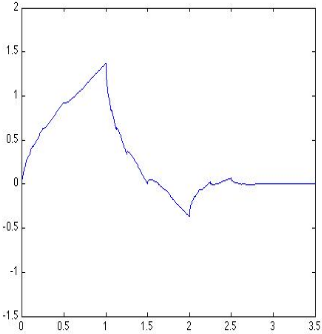
$$\{s_0 = 1, s_1 = 1\}$$

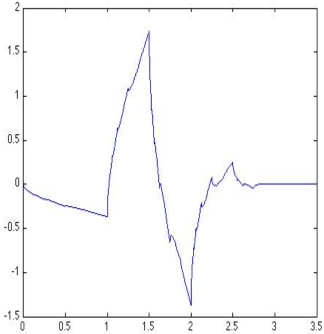




Daubechies Scaling and Wavelet Functions

$$\left\{s_0 = \frac{1+\sqrt{3}}{4}, s_1 = \frac{3+\sqrt{3}}{4}, s_2 = \frac{3-\sqrt{3}}{4}, s_3 = \frac{1-\sqrt{3}}{4}\right\}$$





Orthogonality Conditions

Just as

$$\sum_k s_k s_{k-2j} = 2\delta_{0,j}$$

we will have

$$\sum_k w_k w_{k-2j} = 2\delta_{0,j}$$

Since $W_n \perp V_n$, we have $\langle \psi(x), \phi(x-j) \rangle = 0$, which implies:

$$\sum_k s_k w_{k-2j} = 0 \qquad \forall j \in \mathbb{Z}$$

Discrete Wavelet Transform

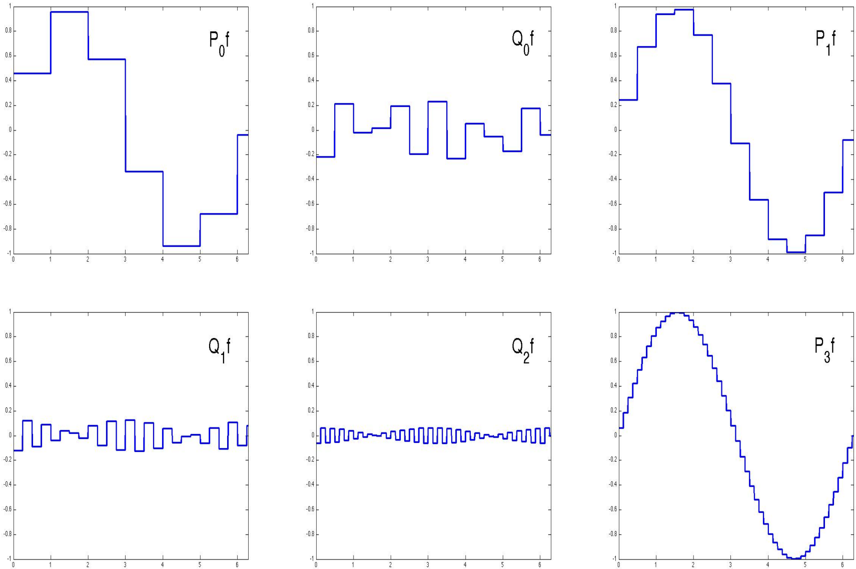
Since

$$V_{n+1} = W_n \oplus V_n = W_n \oplus (W_{n-1} \oplus V_{n-1}) = \ldots = V_l \oplus \left(\bigoplus_{k=l}^{n} W_k \right),$$

$$\forall f \in V_n \qquad f = P_n f = P_l f + \sum_{k=l}^{n-1} Q_k f.$$

So a function in V_n can be expressed as the sum of its approximation at a lower resolution and all of the fine detail at intermediate resolutions.

Fine Detail of a Sine Wave



Let $\ell_{n,k} = \langle f, \phi_{n,k} \rangle$ and $h_{n,k} = \langle f, \psi_{n,k} \rangle$,

$$P_n f = \sum_k \ell_{n,k} \phi_{n,k} = \sum_k \ell_{n-1,k} \phi_{n-1,k} + \sum_k h_{n-1,k} \psi_{n-1,k}.$$

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$$\ell_{n-1,k} = \sum_j s_{j-2k} \ell_{n,j}$$

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With matrix notation,

$$\begin{pmatrix} \vdots \\ \ell_{n-1,0} \\ \ell_{n-1,1} \\ \ell_{n-1,2} \\ \vdots \end{pmatrix} = \begin{pmatrix} & \ddots & & & & & & \\ \dots & s_0 & s_1 & s_2 & s_3 & 0 & 0 & \dots \\ \dots & 0 & 0 & s_0 & s_1 & s_2 & s_3 & \dots \\ \dots & 0 & 0 & 0 & 0 & s_0 & s_1 & \dots \\ & & & & \ddots & & & \end{pmatrix} \begin{pmatrix} \vdots \\ \ell_{n,0} \\ \ell_{n,1} \\ \ell_{n,2} \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \ell_{n-1} \\ h_{n-1} \end{pmatrix} = \begin{pmatrix} S \\ W \end{pmatrix} \left(\ell_n \right)$$

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$$\begin{pmatrix} \ell_{n-1} \\ h_{n-1} \end{pmatrix} = \begin{pmatrix} S \\ W \end{pmatrix} (\ell_n)$$

$$\frac{1}{2} \begin{pmatrix} S \\ W \end{pmatrix}^* \begin{pmatrix} \ell_{n-1} \\ h_{n-1} \end{pmatrix} = (\ell_n)$$

The Implied MRA from S

Suppose

$$V_0 = \{\phi_k(x) = \phi(x-k) : k \in \mathbb{Z}\}$$

and recall

$$\phi\left(\frac{x}{2} - k\right) = \sum_j s_j \phi(x - (j+2k)) = \sum_m s_{m-2k}\phi(x-m)$$

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Define

$$V_{-1} = \left\{\phi_k\left(\frac{x}{2}\right) = \phi\left(\frac{x}{2}-k\right) : k \in \mathbb{Z}\right\}$$

Dealing with Finite Data

Given a sequence $\{f_k\}_{k\in\mathbb{Z}} \in \ell^2$ there is an orthogonal decomposition in $\ell^2 \oplus \ell^2$ defined by

$$\ell_k = \sum_j s_{j-2k} f_j$$

$$h_k = \sum_j w_{j-2k} f_j$$

One method of extending a signal of finite length is to *periodize* the data.

Assume $\{f_k\}$ is real for $k = 0, 1, \ldots, 2n_f - 1$.
Define $F_j = f_k$ when $j = k + 2n_f m$ for some $m \in \mathbb{Z}$.
Apply the previous algorithm to $\{F_j\}_{j\in\mathbb{Z}}$. It will produce periodic ℓ and h.

Simple Periodic Example

4 scaling coefficients and 10 points of data.

$$\begin{pmatrix} \ell_0 \\ h_0 \\ \ell_1 \\ h_1 \\ \ell_2 \\ h_2 \\ \ell_3 \\ h_3 \\ \ell_4 \\ h_4 \end{pmatrix} = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_0 & w_1 & w_2 & w_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_0 & s_1 & s_2 & s_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_0 & w_1 & w_2 & w_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_0 & w_1 & w_2 & w_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & w_0 & w_1 & w_2 & w_3 \\ s_2 & s_3 & 0 & 0 & 0 & 0 & 0 & 0 & s_0 & s_1 \\ w_2 & w_3 & 0 & 0 & 0 & 0 & 0 & 0 & w_0 & w_1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{pmatrix}$$

$$d = Tf$$

MRA on Intervals

Under the assumption that there are $2n_f$ points of data in $\{f_j\}$, $2n+2$ nonzero scaling coefficients, and $n_f > 2n$, we have the following:

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Theorem (Madych, 1997)

There exist orthogonal U and V so that $Q = UTV$ is a banded matrix satisfying $Q^*Q = 2I$.

Under simple conditions (e.g. if the magnitude of the first entry of $Q < \sqrt{2}$) there is a well defined biresolution analysis of $L^2([0, 2n_f])$ such that $V_{-1} \oplus W_{-1} = V_0$.

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(Notice $\frac{1}{\sqrt{2}}Q$ is an orthogonal matrix.)

Proof when n=1

Suppose that there are 4 nonzero scaling coefficients. Recall

$$\sum_k s_k s_{k-2j} = 2\delta_{0,j}$$

and define

$$r_0 = \frac{1}{\sqrt{s_0^2 + s_1^2}} \qquad r_1 = \frac{1}{\sqrt{s_2^2 + s_3^2}}.$$

Then the matrix

$$V = \begin{pmatrix} r_0 s_0 & 0 & 0 & \dots & 0 & 0 & 0 & r_1 s_2 \\ r_0 s_1 & 0 & 0 & \dots & 0 & 0 & 0 & r_1 s_3 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ & & & \ddots & \ddots & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}$$

is orthogonal.

Now $TV =$

$$\begin{pmatrix} s_0 & s_1 & s_2 & s_3 & 0 & 0 & \dots & 0 \\ w_0 & w_1 & w_2 & w_3 & 0 & 0 & \dots & 0 \\ 0 & 0 & s_0 & s_1 & s_2 & s_3 & \dots & 0 \\ 0 & 0 & w_0 & w_1 & w_2 & w_3 & \dots & 0 \\ & & & \ddots & \ddots & & & \\ 0 & 0 & \dots & 0 & s_0 & s_1 & s_2 & s_3 \\ 0 & 0 & \dots & 0 & w_0 & w_1 & w_2 & w_3 \\ s_2 & s_3 & 0 & 0 & \dots & 0 & s_0 & s_1 \\ w_2 & w_3 & 0 & 0 & \dots & 0 & w_0 & w_1 \end{pmatrix} \begin{pmatrix} r_0 s_0 & 0 & 0 & \dots & 0 & 0 & 0 & r_1 s_2 \\ r_0 s_1 & 0 & 0 & \dots & 0 & 0 & 0 & r_1 s_3 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ & & & \ddots & \ddots & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sum_k s_k w_{k-2j} = 0 \text{ and } \sum_k s_k s_{k-2j} = 2\delta_{0,j} \implies TV \text{ is banded.}$$

Choosing arbitrary orthogonal matrices U_x and U_y and defining

$$U = \begin{pmatrix} U_x & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U_y \end{pmatrix}$$

will allow

$$Q = UTV$$

to be orthogonal (after multiplying by $\frac{1}{\sqrt{2}}$) while giving some control over the actual entries of Q.

Let

$$\begin{pmatrix} M \\ N \end{pmatrix} = U \begin{pmatrix} S \\ W \end{pmatrix} V$$

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If

$$M = \begin{pmatrix} a_0 & a_1 & a_2 & 0 & 0 & \dots & 0 \\ 0 & s_0 & s_1 & s_2 & s_3 & \dots & 0 \\ 0 & 0 & 0 & s_0 & s_1 & \dots & 0 \\ & & \ddots & \ddots & & & \\ 0 & \dots & s_0 & s_1 & s_2 & s_3 & 0 \\ 0 & \dots & 0 & 0 & b_0 & b_1 & b_2 \end{pmatrix}$$

$V_0 \subset L^2([0, 2n_f])$ consists of $\{\phi_A(x), \phi_j(x), \phi_B(x)\}$ with $j = 0, \dots, 2n_f - 3$.

If V_0 consists of $\{\phi_A(x), \phi_j(x), \phi_B(x)\}$, define

$$V_{-1} = \left\{\phi_A\left(\frac{x}{2}\right), \phi_j\left(\frac{x}{2}\right), \phi_B\left(\frac{x}{2} + n_f\right)\right\} \quad j = 0, \ldots, n_f - 3$$

where

$$\phi_A\left(\frac{x}{2}\right) = a_0\phi_A(x) + a_1\phi(x) + a_2\phi(x-1)$$

$$\phi\left(\frac{x}{2} - j\right) = s_0\phi(x-2j) + s_1\phi(x-2j-1) + s_2\phi(x-2j-2) + s_3\phi(x-2j-3)$$

$$\phi_B\left(\frac{x}{2} + n_f\right) = b_0\phi(x - 2n_f + 4) + b_1\phi(x - 2n_f + 3) + b_2\phi_B(x)$$

It can be shown that this MRA is well-defined provided $|a_0| < \sqrt{2}$ and $|b_2| < \sqrt{2}$. $\square$

Why is this the definition of ϕ_B?

$$\phi_B\left(\frac{x}{2}+n_f\right)=b_0\phi(x-2n_f+4)+b_1\phi(x-2n_f+3)+b_2\phi_B(x)$$

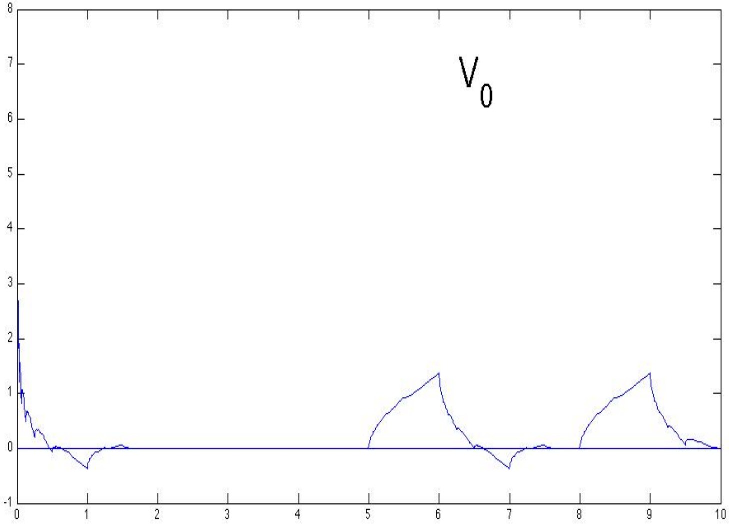
Consider that on a finite interval $[0,2n_f]$ when you dilate a function which terminates at the right endpoint, the support is no longer in the interval. It makes more sense to interpret dilation *restricted to the interval*. Define V_{-1} then, as consisting of

$$V_{-1}=\left\{\phi_A\left(\frac{x}{2}\right),\phi_j\left(\frac{x}{2}\right),\phi_B\left(\frac{x}{2}+n_f\right)\right\}\quad j=0,\ldots,n_f-3$$

and note now that the definition of $\phi_B\left(\frac{x}{2}+n_f\right)$ has not changed, it is still $M\left(\phi_j\right)$ as required. This explains the odd shift in the definition.
It is easier to consider $\tilde{\phi}_B(x)=\phi_B(x+2n_f)$. In this case we simply add $2n_f$ to the argument in the equation defining ϕ_B and observe that it is now equivalent to

$$\tilde{\phi}_B\left(\frac{x}{2}\right)=b_0\phi(x+4)+b_1\phi(x+3)+b_2\tilde{\phi}_B(x)$$

Example of MRA on $L^2([0, 10])$



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A necessary condition is that

$$\sum_k s_k \tilde{s}_{k-2j} = 2\delta_{0,j}$$

The projections onto the spaces V_n and $\widetilde{V}_n$ are given by

$$P_n f = \sum_k \langle f, \tilde{\phi}_{n,k} \rangle \phi_{n,k} \qquad \widetilde{P}_n f = \sum_k \langle f, \phi_{n,k} \rangle \tilde{\phi}_{n,k}$$

$$V_n \oplus W_n = V_{n+1} \qquad \widetilde{V}_n \oplus \widetilde{W}_n = \widetilde{V}_{n+1}$$

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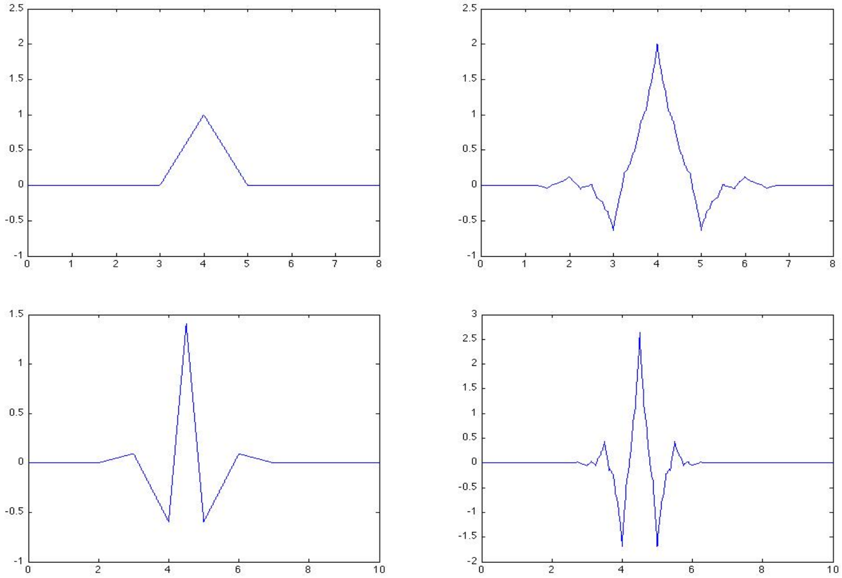
The *biorthogonality conditions* are:

$$\sum_k s_k \tilde{s}_{k-2j} = 2\delta_{0,j}$$

$$\sum_k w_k \tilde{w}_{k-2j} = 2\delta_{0,j}$$

$$\sum_k s_k \tilde{w}_{k-2j} = \sum_k w_k \tilde{s}_{k-2j} = 0$$

Biorthogonal Example



Biorthogonal Setup

$$\{s_{-1}, s_0, s_1\} \to \{s_3, s_4, s_5\}$$

$$\{\tilde{\tilde{s}}_{-4}, \tilde{s}_{-3}, \tilde{s}_{-2}, \tilde{s}_{-1}, \tilde{s}_0, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\} \to \{\tilde{s}_0, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_6, \tilde{s}_7, \tilde{s}_8\}$$

$$\{0, 0, 0, s_3, s_4, s_5, 0, 0, 0, 0\}$$

$$\{\tilde{s}_0, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_6, \tilde{s}_7, \tilde{s}_8, 0\}$$

T consists of rows of scaling and wavelet coefficients that look like

$$\{0, 0, 0, s_3, s_4, s_5, 0, 0, 0, 0\}$$

$$\{\tilde{s}_8, -\tilde{s}_7, \tilde{s}_6, -\tilde{s}_5, \tilde{s}_4, -\tilde{s}_3, \tilde{s}_2, -\tilde{s}_1, \tilde{s}_0, 0\}$$

with $\widetilde{T}$ defined similarly.

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with $\widetilde{T}$ defined similarly.

Because of the biorthogonality conditions, $\left(\frac{1}{\sqrt{2}} T, \frac{1}{\sqrt{2}} \widetilde{T}\right)$ are biorthogonal, i.e.

$$\frac{1}{2} T \widetilde{T}^* = I$$

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Because of the biorthogonality conditions, $\left(\frac{1}{\sqrt{2}} T, \frac{1}{\sqrt{2}} \widetilde{T}\right)$ are biorthogonal, i.e.

$$\frac{1}{2} T \widetilde{T}^* = I$$

In this case we require biorthogonal $\left(U, \widetilde{U}\right)$, $\left(V, \widetilde{V}\right)$ so that $Q = \widetilde{U} T \widetilde{V}$, $\widetilde{Q} = U \widetilde{T} V$ are banded matrices satisfying

$$Q \widetilde{Q}^* = 2I$$

$$T = \begin{pmatrix}
s_0 & s_1 & s_2 & s_3 & \dots & s_{2n+1} & 0 & \dots & 0 & 0 \\
w_0 & w_1 & w_2 & w_3 & \dots & w_{2n+1} & 0 & \dots & 0 & 0 \\
0 & 0 & s_0 & s_1 & s_2 & s_3 & \dots & s_{2n+1} & 0 & \vdots \\
0 & 0 & w_0 & w_1 & w_2 & w_3 & \dots & w_{2n+1} & 0 & \vdots \\
\vdots & \vdots & & & \ddots & \ddots & & & & \vdots \\
0 & 0 & 0 & \dots & s_0 & s_1 & s_2 & \dots & s_{2n} & s_{2n+1} \\
0 & 0 & 0 & \dots & w_0 & w_1 & w_2 & \dots & w_{2n} & w_{2n+1} \\
s_{2n} & s_{2n+1} & 0 & 0 & \dots & \ddots & \ddots & & s_{2n-2} & s_{2n-1} \\
w_{2n} & 2_{2n+1} & 0 & 0 & \dots & & \ddots & \ddots & w_{2n-2} & w_{2n-1} \\
\vdots & & \ddots & & & & & \ddots & \vdots & \vdots \\
s_2 & s_3 & \dots & s_{2n+1} & 0 & \dots & \dots & 0 & s_0 & s_1 \\
w_2 & w_3 & \dots & w_{2n+1} & 0 & \dots & \dots & 0 & w_0 & w_1
\end{pmatrix}$$

T is a $2n_f \times 2n_f$ matrix with $2n+2$ diagonals and a block matrix $(2n \times 2n)$ in the lower left corner.

To simplify the appearance, let

$$T_k = \begin{pmatrix} s_{2k} & s_{2k+1} \\ w_{2k} & w_{2k+1} \end{pmatrix}$$

$$T_a = \begin{pmatrix} T_0 & T_1 & \dots & T_{n-1} \\ 0 & T_0 & \dots & T_{n-2} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & T_0 \end{pmatrix}$$

$$T_b = \begin{pmatrix} T_n & 0 & 0 & 0 \\ T_{n-1} & T_n & 0 & 0 \\ \vdots & & \ddots & 0 \\ T_1 & T_2 & \dots & T_n \end{pmatrix}$$

Then

$$T = \begin{pmatrix} T_a & T_b & 0_{2n\times 2k} \\ 0_{2k\times 2n} & T_c & \\ T_b & 0_{2n\times 2k} & T_a \end{pmatrix}$$

Where $n_f = 2n + k$, and T_c is a $2k \times 2(n+k)$ matrix whose rows are simply shifts of $\begin{pmatrix} T_0 & T_1 & \dots & T_n \end{pmatrix}$:

$$T_c = \begin{pmatrix} T_0 & T_1 & \dots & T_n & 0_{2\times 2} & \dots & 0_{2\times 2} \\ 0_{2\times 2} & T_0 & T_1 & \dots & T_n & 0_{2\times 2} & \dots \\ \vdots & & & \ddots & \ddots & & \\ 0_{2\times 2} & \dots & T_0 & T_1 & \dots & T_{n-1} & T_n \end{pmatrix}$$

For $k = 0, ..., n$, defining $S_k = \begin{pmatrix} s_{2k} & s_{2k+1} \end{pmatrix}$, W_k, $\widetilde{S}_k$, and $\widetilde{W}_k$ similarly, and S_a, S_b, $\widetilde{S}_a$, $\widetilde{S}_b$, W_a, W_b, $\widetilde{W}_a$, $\widetilde{W}_b$ in the same manner as T, we have the following:

Lemma

$$T_a \widetilde{S}_b^* = 0$$

Proof:

$$T_a \widetilde{S}_b^* = \begin{pmatrix} T_0 & T_1 & T_2 & \dots & T_{n-1} \\ 0 & T_0 & T_1 & \dots & T_{n-2} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & T_0 & T_1 \\ 0 & 0 & 0 & \dots & T_0 \end{pmatrix} \begin{pmatrix} \widetilde{S}_n^* & \widetilde{S}_{n-1}^* & \dots & \widetilde{S}_1^* \\ 0 & \widetilde{S}_n^* & \dots & \widetilde{S}_2^* \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \widetilde{S}_n^* & \widetilde{S}_{n-1}^* \\ 0 & 0 & \dots & \widetilde{S}_n^* \end{pmatrix}$$

$$T_a\widetilde{S}_b^* = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 & \dots & s_{2n-2} & s_{2n-1} \\ w_0 & w_1 & w_2 & w_3 & \dots & w_{2n-2} & w_{2n-1} \\ 0 & 0 & s_0 & s_1 & \dots & s_{2n-4} & s_{2n-3} \\ 0 & 0 & w_0 & w_1 & \dots & w_{2n-4} & w_{2n-3} \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & s_0 & s_1 \\ 0 & 0 & 0 & 0 & \dots & w_0 & w_1 \end{pmatrix} \begin{pmatrix} \tilde{s}_{2n} & \tilde{s}_{2n-2} & \dots & \tilde{s}_2 \\ \tilde{s}_{2n+1} & \tilde{s}_{2n-1} & \dots & \tilde{s}_3 \\ 0 & \tilde{s}_{2n} & \dots & \tilde{s}_4 \\ 0 & \tilde{s}_{2n+1} & \dots & \tilde{s}_5 \\ 0 & 0 & \dots & \tilde{s}_6 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \tilde{s}_{2n} \\ 0 & 0 & \dots & \tilde{s}_{2n+1} \end{pmatrix}$$

□

Similarly,

$$T_a\widetilde{W}_b^* = \widetilde{T}_a S_b^* = \widetilde{T}_a W_b^* = 0$$

and

$$\widetilde{T}_b S_a^* = \widetilde{T}_b W_a^* = T_b\widetilde{S}_a^* = T_b\widetilde{W}_a^* = 0$$

Theorem

Given T and $\widetilde{T}$ biorthogonal, there exist biorthogonal pairs of matrices $\left(U, \widetilde{U}\right)$ and $\left(V, \widetilde{V}\right)$ such that $Q = \widetilde{U} T \widetilde{V}$, $\widetilde{Q} = U \widetilde{T} V$ are biorthogonal and banded.

Proof:

Choose $\left(U, \widetilde{U}\right)$ an arbitrary biorthogonal pair, and define

$$V = \begin{pmatrix} (R_a S_a)^* & 0_{2n,2n_f-2n} & (R_b S_b)^* \\ 0_{2n_f-2n,n} & I_{2n_f-2n,2n_f-2n} & 0_{2n_f-2n,n} \end{pmatrix}$$

$$\widetilde{V} = \begin{pmatrix} \widetilde{W}_a^* & 0_{2n,2n_f-2n} & \widetilde{W}_b^* \\ 0_{2n_f-2n,n} & I_{2n_f-2n,2n_f-2n} & 0_{2n_f-2n,n} \end{pmatrix}$$

$\left(V, \widetilde{V}\right)$ must be biorthogonal. This implies $R_a S_a \widetilde{W}_a^* = I$ and $R_b S_b \widetilde{W}_b^* = I$, which defines R_a and R_b (provided $S_a \widetilde{W}_a^*$ and $S_b \widetilde{W}_b^*$ are invertible.)

$$T\widetilde{V} = \begin{pmatrix} T_a & T_b & 0_{2n\times 2k} \\ 0_{2k\times 2n} & T_c & \\ T_b & 0_{2n\times 2k} & T_a \end{pmatrix} \begin{pmatrix} \widetilde{W}_a^* & 0_{2n,2n_f-2n} & \widetilde{W}_b^* \\ 0_{2n_f-2n,n} & I_{2n_f-2n,2n_f-2n} & 0_{2n_f-2n,n} \end{pmatrix}$$

$$= \begin{pmatrix} T_a\widetilde{W}_a^* & T_b & 0 & T_a\widetilde{W}_b^* \\ 0 & T_c & & 0 \\ T_b\widetilde{W}_a^* & 0 & T_a & T_b\widetilde{W}_b^* \end{pmatrix} = \begin{pmatrix} T_a\widetilde{W}_a^* & T_b & 0 & 0 \\ 0 & T_c & & 0 \\ 0 & 0 & T_a & T_b\widetilde{W}_b^* \end{pmatrix}$$

A similar result holds for $\widetilde{T}V$. □

Some questions/goals yet to be considered:

- Address continuity and regularity of the MRA of $L^2([0, 2n_f])$
- Extend the concept of biorthogonal MRA to dual spaces beyond L^2
- Can scaling functions not of compact support be used somehow?
- How does one handle *signals* of odd lengths without zero padding (adding zeroes at the end of the data)?

References

Thanks!

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