FINITE BIORTHOGONAL TRANSFORMS
AND
MULTIRESOLUTION ANALYSES ON INTERVALS

Background

Suppose scaling coefficients $\{s_k\}$, $\{\tilde{s}_k\}$ are given such that ϕ and $\tilde{\phi}$ are two compactly supported functions in $L^2(\mathbb{R})$ which satisfy the *scaling equation*

$$\phi(x) = \sqrt{2}\sum_k s_k \phi(2x-k), \quad \tilde{\phi}(x) = \sqrt{2}\sum_k \tilde{s}_k \tilde{\phi}(2x-k),$$

that ϕ and $\tilde{\phi}$ are biorthogonal

$$\left\langle \phi(x), \tilde{\phi}(x-n) \right\rangle = \delta_{0n},$$

and that their integer shifts form Riesz bases for two subspaces of $L^2(\mathbb{R})$

$$V_0 = \text{Linear Span of } \{\phi(x-k) : k \in \mathbb{Z}\},$$
$$\widetilde{V}_0 = \text{Linear Span of } \{\tilde{\phi}(x-k) : k \in \mathbb{Z}\}.$$

Multiresolution analyses (MRAs) of $L^2(\mathbb{R})$ are generated by imposing the conditions

$$f(x) \in V_k \iff f(2x) \in V_{k+1}, \quad f(x) \in \widetilde{V}_k \iff f(2x) \in \widetilde{V}_{k+1},$$

(It is also assumed that $\bigcup V_n$ and $\bigcup \widetilde{V}_n$ are dense in $L^2(\mathbb{R})$.)

Let

$$\phi_{n,k} = 2^{n/2}\phi(2^n x - k), \quad \tilde{\phi}_{n,k} = 2^{n/2}\tilde{\phi}(2^n x - k).$$

Define the projections from L^2 onto V_n and $\widetilde{V}_n$ by

$$P_n(f) = \sum_k \langle f(x), \tilde{\phi}_{n,k}(x)\rangle \phi_{n,k}(x), \quad \widetilde{P}_n(f) = \sum_k \langle f(x), \phi_{n,k}(x)\rangle \tilde{\phi}_{n,k}(x).$$

Letting $Q_n = P_{n+1} - P_n$ and $\widetilde{Q}_n = \widetilde{P}_{n+1} - \widetilde{P}_n$, we define the spaces W_n and $\widetilde{W}_n$ as the range of Q_n and $\widetilde{Q}_n$, respectively. These spaces satisfy

$$V_n \bigoplus W_n = V_{n+1}, \quad \widetilde{V}_n \bigoplus \widetilde{W}_n = \widetilde{V}_{n+1}.$$

These are not necessarily orthogonal sums, however the biorthogonality implies

$$W_n \perp \widetilde{V}_n, \quad \widetilde{W}_n \perp V_n.$$

A fundamental result is that $L^2(\mathbb{R}) = \bigoplus W_n = \bigoplus \widetilde{W}_n$, and that there is an extremely simple way to define these spaces W_n and $\widetilde{W}_n$. There exist wavelet functions

$$\psi(x) = \sqrt{2}\sum_k w_k \phi(2x-k), \quad \tilde{\psi}(x) = \sqrt{2}\sum_k \tilde{w}_k \tilde{\phi}(2x-k)$$

where

$$w_k = (-1)^k \tilde{s}_{N-k}, \quad \tilde{w}_k = (-1)^k s_{N-k} \quad \text{for fixed odd N}$$

such that

$$\begin{aligned} W_n &= \text{The linear span of } \{2^{n/2}\psi(2^n x - k) : k \in \mathbb{Z}\}, \\ \widetilde{W}_n &= \text{The linear span of } \{2^{n/2}\tilde{\psi}(2^n x - k) : k \in \mathbb{Z}\}. \end{aligned}$$

Furthermore, defining bi-infinite matrices $S^\infty = \left(S^\infty_{j,k}\right) = (s_{j-2k})$, $W^\infty = \left(W^\infty_{jk}\right) = (w_{j-2k})$, $\widetilde{S}^\infty = (\widetilde{S}^\infty_{jk}) = (\tilde{s}_{j-2k})$, $\widetilde{W}^\infty = (\widetilde{W}^\infty_{jk}) = (\tilde{w}_{j-2k})$, produces an invertible map $T : l^2 \to l^2 \times l^2$ via the discrete wavelet transform. The discrete wavelet transform applied to a signal $f \in l^2(\mathbb{R})$ is defined by

$$Tf = \begin{pmatrix} S \\ W \end{pmatrix} f = \begin{pmatrix} \ell \\ h \end{pmatrix},$$

and has exact reconstruction

$$f = \begin{pmatrix} \widetilde{S} \\ \widetilde{W} \end{pmatrix}^* \begin{pmatrix} S \\ W \end{pmatrix} f = \begin{pmatrix} \widetilde{S} \\ \widetilde{W} \end{pmatrix}^* \begin{pmatrix} \ell \\ h \end{pmatrix}.$$

To apply this theory to a *finite* length $f = \{f_k\}_{k=0}^{2n_f-1}$ a natural approach would be to first *periodize* the data. The resulting decompositions ℓ and h are then periodic. Considering one period of ℓ and h is equivalent to applying certain finite matrices to the original finite signal f. We denote these matrices S, W, $\widetilde{S}$, and $\widetilde{W}$. For instance,

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_4 & s_5 & 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 & s_5 & 0 & 0 & 0 & 0 & s_0 & s_1 \end{pmatrix}.$$

This transformation will intertwine the data at the beginning and end of f, which is undesirable. Theorem 1 addresses this.

Given a finite number of scaling coefficients $\{s_k\}_{k=0}^{2n+1}$ and $\{\tilde{s}_k\}_{k=0}^{2n+1}$ producing biorthogonal scaling functions ϕ and $\tilde{\phi}$, we let $T = \begin{pmatrix} S \\ W \end{pmatrix}$, $\widetilde{T} = \begin{pmatrix} \widetilde{S} \\ \widetilde{W} \end{pmatrix}$, where S, W, $\widetilde{S}$, and $\widetilde{W}$ are the finite "periodized" matrices just described.

A pair of square matrices $\left(M, \widetilde{M}\right)$ are called *biorthogonal* if $\widetilde{M}^* M = \widetilde{M} M^* = I$.
$\left(T, \widetilde{T}\right)$ form a biorthogonal pair.

Theorem 1. *Given the matrices T and $\widetilde{T}$, there exist biorthogonal pairs of matrices $\left(U, \widetilde{U}\right)$ and $\left(V, \widetilde{V}\right)$ such that $Q = \widetilde{U} T \widetilde{V}$, $\widetilde{Q} = U \widetilde{T} V$ are biorthogonal and consist of banded block matrices.*

V and $\widetilde{V}$ make use of the biorthogonality of the scaling coefficients and are defined so that $T\widetilde{V}$ and $\widetilde{T}V$ are banded. U and $\widetilde{U}$ can be used to modify the entries in the upper-left and lower-right sub-blocks of Q and $\widetilde{Q}$.

Theorem 1 provides us with *banded* coefficient matrices, which can be used to define a multiresolution analysis. Letting $Q = \begin{pmatrix} M \\ N \end{pmatrix}$, $\widetilde{Q} = \begin{pmatrix} \widetilde{M} \\ \widetilde{N} \end{pmatrix}$, we have banded matrices

$$M = \left(\begin{array}{c|ccc|c} A_0 & A_1 & A_2 & 0 & 0 \\ \hline 0 & & S_c & & 0 \\ \hline 0 & 0 & B_0 & B_1 & B_2 \end{array}\right), \quad \widetilde{M} = \left(\begin{array}{c|ccc|c} \widetilde{A}_0 & \widetilde{A}_1 & \widetilde{A}_2 & 0 & 0 \\ \hline 0 & & \widetilde{S}_c & & 0 \\ \hline 0 & 0 & \widetilde{B}_0 & \widetilde{B}_1 & \widetilde{B}_2 \end{array}\right),$$

where S_c is a block matrix of interior rows of scaling coefficients $\{s_k\}$ that were unaltered by the transform. A_0, A_1, A_2, and B_0, B_1, B_2 are $n \times n$ block matrices.

The rows of these modified matrices can be regarded as scaling coefficients and used to construct a multiresolution analysis over a finite interval.

Define

$$V_0 = \text{The linear span of } \{\phi_j(x)\}_{j=0}^{n_f-1}$$

where

$$\phi_j(x) = \begin{cases} \phi_{A,j}(x) & j = 0, \dots, n-1 \\ \phi(x-j-n) & j = n, \dots, n_f - n - 1 \\ \phi_{B,j-n_f+n}(x) & j = n_f - n, \dots, n_f - 1 \end{cases}$$

with $\widetilde{V}_0$ defined similarly. The "interior" functions are simply integer shifts of $\phi(x)$, the original scaling function. The functions $\phi_{A,j}$ and $\phi_{B,j}$ are left and right boundary functions defined via the first or last n rows of M, respectively. The collection is defined by the scaling relation

$$\Phi_0(x) = (\phi_j(x)) = \sqrt{2} M \Phi_1(2x), \quad \widetilde{\Phi}_0(x) = \left(\tilde{\phi}_j(x)\right) = \sqrt{2}\widetilde{M}\widetilde{\Phi}_1(2x), \tag{1}$$

where Φ_0 is a column vector (of length n_f) whose j^{th} component is the function ϕ_j, and Φ_1 is a column vector (of length $2n_f$) which also contains the functions ϕ_j, but includes more interior functions (more shifts of the scaling function ϕ). For instance, in the case of $n = 1$ we would have

$$\phi_A(x) = \sqrt{2}\left(a_0 \phi_A(2x) + a_1 \phi(2x) + a_2 \phi(2x-1)\right).$$

These collections V_0 and $\widetilde{V}_0$ are well-defined and biorthogonal if $|A_0|, |\widetilde{A}_0|, |B_2|, |\widetilde{B}_2| < 1$. (Where $|M|$ can be the max-norm of the entries from a matrix M, or the more typical operator norm (spectral norm) corresponding to the 2-norm for vectors.)

Theorem 2. *There exist unique solutions to (1), $\{\phi_j\}$ and $\{\tilde{\phi}_j\}$, which are compactly supported and satisfy $\langle \phi_j, \tilde{\phi}_k \rangle = \delta_{j,k}$ provided that $|A_0|$, $|\widetilde{A}_0|$, $|B_2|$, and $|\widetilde{B}_2| < 1$.*

We define W_0 as the linear span of the components of Ψ_0, and $\widetilde{W}_0$ as the linear span of the components of $\widetilde{\Psi}_0$ where

$$\Psi_0(x) = (\psi_j(x)) = \sqrt{2} N \Phi_1(2x), \quad \widetilde{\Psi}_0(x) = \left(\tilde{\psi}_j(x)\right) = \sqrt{2}\widetilde{N}\widetilde{\Phi}_1(2x).$$

V_1 is defined as the linear span of the components of $\Phi_1(2x)$, and similarly for $\widetilde{V}_1$. This produces multiresolution analyses over $[0, 2n_f]$ consisting of finite-dimensional subspaces.

Theorem 3. *Given biorthogonal scaling functions $\phi(x)$ and $\tilde{\phi}(x)$ supported in a finite interval $[0, 2n_f]$, we have $V_0 \bigoplus W_0 = V_1$ and $\widetilde{V}_0 \bigoplus \widetilde{W}_0 = \widetilde{V}_1$ (as oblique sums) as well as $V_0 \perp \widetilde{W}_0$ and $\widetilde{V}_0 \perp W_0$.*

Now that we have defined the scaling and wavelet functions, their regularity and approximation properties can be explored. Let C^α denote functions which are Hölder continuous of order α.

Theorem 4. *If $\phi \in C^\alpha$ then $\phi_{A,j}(x)$ is C^α for $x > 0$.*
If $\phi \in C^\alpha$ and $|A_0| < 2^{-\alpha-1/2}$, then $\phi_{A,j}(x)$ is C^α at 0.

Theorem 5. *If $|A_0| = \frac{1}{\sqrt{2}}$ and ϕ is Lipschitz, then Φ_A is bounded.*